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THE ALPERIN AND DADE CONJECTURES FOR THE SIMPLE CONWAY'S THIRD GROUP

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ABSTRACT

This paper is part of a program to study Alperin's weight conjecture and Dade's conjecture on counting ordinary characters in blocks for several finite groups. The classifications of radical subgroups and certain radical chains and their local structures of the simple Conway's third group have been obtained by using the computer algebra system CAYLEY. The Alperin weight conjecture and the Dade final conjecture have been confirmed for the group.

Introduction

Let G be a finite group, p a prime and B a p-block of G. Alperin in [1] conjectured that the number of B-weights should equal the number of irreducible Brauer characters of B. Dade in [7] has presented a conjecture exhibiting the number of ordinary irreducible characters of a fixed height in B, in terms of an alternating sum of similar integers for p-blocks of some local subgroups of the group G. By Dade [7], his final conjecture needs only to be verified for finite non-abelian simple groups and is equivalent to the ordinary conjecture whenever a finite group has a trivial Schur multiplier and outer automorphism group. In this paper we verify the Alperin weight conjecture and the Dade ordinary conjecture, and so the final one, for the simple Conway's third group.

Most of the calculations were carried out using the CAYLEY computer system [4]. In Section 1, we fix our notation and state the two conjectures. In Section 2, we classify radical subgroups, determine their local structures and verify the

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Alperin weight conjecture. In Section 3, we do some cancellations in the alternating sum when $p = 2$ or 3, and then determine radical chains and their local structures. In the last Section, we verify the Dade conjecture.

1. The Alperin and Dade conjectures

Let R be a p-subgroup of a finite group G. Then R is **radical** if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a B-**weight** if φ has p-defect 0 and $B(\varphi)^G = B$ (in the sense of Brauer), where $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and $\ell(B)$ the number of irreducible Brauer characters of B. Alperin conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$. If a defect group D of B is cyclic, the Alperin conjecture follows by Theorem 9.1 of [7]. Thus we may suppose D is non-cyclic.

Given a p-subgroup chain

$$C\colon P_0 < P_1 < \cdots < P_n \tag{1.1}$$

of a finite group G, define $|C| = n$, $C_k\colon P_0 < P_1 < \cdots < P_k$, $C(C) = C_G(P_n)$, and

$$N(C) = N_G(C) = N(P_0) \cap N_G(P_1) \cap \cdots \cap N_G(P_n). \tag{1.2}$$

The chain C is said to be **radical** if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$ and (b) $P_k = O_p(N(C_k))$

for $1 \le k \le n$. Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. For $B \in \mathrm{Blk}(G)$ and integer $d \ge 0$, let $\mathrm{k}(N(C), B, d)$ be the number of characters in the set

$$\mathrm{Irr}(N(C), B, d) = \{\psi \in \mathrm{Irr}(N(C))\colon B(\psi)^G = B,\ \mathrm{d}(\psi) = d\}, \tag{1.3}$$

where $\mathrm{d}(\psi)$ is the defect of ψ (see [7, (5.5)] for the definition). Dade in [7] gives the following conjecture.

Ordinary conjecture: *If $O_p(G) = 1$ and B is a p-block of G with defect $\mathrm{d}(B) > 0$, then for any integer $d \ge 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|}\,\mathrm{k}(N(C), B, d) = 0, \tag{1.4}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits in $\mathcal{R}$.

2. Radical subgroups and weights

Let $\Phi(G,p)$ be a set of representatives for conjugacy classes of radical subgroups of G. For $H, K \leq G$, we write $H \leq_G K$ if $x^{-1}Hx \leq K$; and write $H \in_G \Phi(G,p)$ if $x^{-1}Hx \in \Phi(G,p)$ for some $x \in G$. We shall follow the notation of [6]. In particular, $p_+^{1+2\gamma}$ is an extra special group of order $p^{1+2\gamma}$ with exponent p or plus type according as p is odd or even. If X and Y are groups, we use $X.Y$ and $X:Y$ to denote an extension and a split extension of X by Y, respectively. Given $n \in \mathbb{N}$, we use E_{p^n} or simply p^n to denote the elementary abelian group of order p^n, $\mathbb{Z}_n$ or simply n to denote the cyclic group of order n, and D_{2n} to denote the dihedral group of order $2n$.

Let G be the simple Conway's third group Co_3. Then

$$|G| = 2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 23,$$

and we may suppose $p \in \{2,3,5\}$, since both conjectures hold for a block with a cyclic defect group by [7].

We denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H. Given $R \in \Phi(G,p)$, let $C(R) = C_G(R)$ and $N = N_G(R)$. If $B_0 = B_0(G)$ is the principal p-block of G, then by [3, (1.3)],

$$\mathcal{W}(B_0) = \sum_R |\mathrm{Irr}^0(N/C(R)R)|, \tag{2.1}$$

where R runs over the set $\Phi(G,p)$ such that the p-part $|C(R)R/R|_p$ of $|C(R)R/R|$ is 1. The character table of $N/C(R)R$ can be created by CAYLEY, so that we can find the number $|\mathrm{Irr}^0(N/C(R)R)|$.

In the tables of Propositions (2A), (2B) and (2C), if the p-part $|C(R)R/R|_p$ is not 1, then by (2.1), there exist no B_0-weights of the form (R,φ), so that we omit the number $|\mathrm{Irr}^0(N/C(R)R)|$.

(2A): *The non-trivial radical* 5*-subgroups* R *of* Co_3 *(up to conjugacy) are*

R	$C(R)$	N	$\lvert\mathrm{Irr}^0(N/C(R)R)\rvert$
5	$5 \times A_5$	$(5 \times A_5).4$	
5_+^{1+2}	5	$5_+^{1+2}.(3 \times 8).2$	18.

Proof: The proof of (2A) follows by Lemmas 5.7 and 5.14 of [9].

The calculations in the Propositions (2B) and (2C) are carried out using the CAYLEY computer system. The approach using CAYLEY to classify radical p-subgroup classes is explained in [3]. We first choose a Sylow p-subgroup S (using

CAYLEY), and then calculate the subgroup lattice of S. For each subgroup R in the lattice, we calculate the normalizer $N(R)$ of R in G and the normal subgroup classes of $N(R)$. If the largest order of the p-subgroups in the normal subgroup classes of $N(R)$ is the same as the order of R, then R is a radical subgroup of G. For a p-group H, CAYLEY can test whether or not H is abelian (using the centre of the subgroup) and elementary abelian. In the Propositions (2B) and (2C), we also calculate the radical p-subgroups of some maximal subgroups; these radical subgroups will be used in Section 3 to classify the radical p-chains.

(2B): *The non-trivial radical* 3*-subgroups R of* Co_3 *(up to conjugacy) are*

R	$C(R)$	N	$\lvert \mathrm{Irr}^0(N/C(R)R)\rvert$
3	$3 \times L_2(8):3$	$S_3 \times L_2(8):3$	
3_+^{1+4}	3	$3_+^{1+4}{:}\,4S_6$	4
3^5	3^5	$3^5{:}\,(2 \times M_{11})$	2
S	3	$S:(2 \times SD_{2^4})$	14,

where $S \in \mathrm{Syl}_3(\mathrm{Co}_3)$ and SD_{2^4} is a semidihedral group of order 2^4.

Proof: Given $i \in \{1,2,3\}$, let M_i be a maximal subgroup of $G = \mathrm{Co}_3$ such that $M_1 \simeq S_3 \times L_2(8){:}\,3$, $M_2 \simeq 3_+^{1+4}:4S_6$ and $M_3 \simeq 3^5{:}\,(2 \times M_{11})$ (see [6, p. 134]). If T is an elementary 3-subgroup of G, then by [9, p. 73], $N(T) \leq M_i$ for some i. If R is a non-trivial radical 3-subgroup of G, then $\Omega_1(Z(R))$ is elementary abelian and $N(R) \leq N(\Omega_1(Z(R)))$, since $\Omega_1(Z(R))$ is a characteristic subgroup of R. Thus we may suppose $R \in \Phi(M_i, 3)$ for some i and $N(R) = N_{M_i}(R)$.

Using CAYLEY, we can calculate the normalizers of all the 3-subgroups of G (see the remark before Proposition (2B)). If M is a normalizer of a subgroup of order 3 such that M has the same composition factors, which can be obtained by CAYLEY, as that of M_1, then M is conjugate to M_1. Similarly, since M_2 and M_3 are normalizers of some 3-subgroups, we can easily identify them from the normalizers of 3-subgroups of G.

Let $M = M_1$, $3 = O_3(M_1)$ and $S' \in \mathrm{Syl}_3(M_1)$. Apply the approach described before (2B) to M. We have that M has two radical 3-subgroups with orders 3 and $|S'|$. Note that a Sylow 3-subgroup is always a radical subgroup and $O_3(M)$ is a subgroup of each radical 3-subgroup of M (cf. [11, Lemma 2.1]). Thus we may suppose

$$\Phi(S_3 \times L_2(8){:}\,3, 3) = \{3, S'\}.$$

Using CAYLEY, we find that the normalizers $N_{M_1}(S')$ and $N_G(S')$ have different orders, so that $N_{M_1}(S') \neq N(S')$. Since M is maximal in G, $N_G(3) = M = N_M(3)$ and so we may suppose $3 \in \Phi(G, 3)$.

Let $3_+^{1+4} = O_3(M_2)$ and $3^5 = O_3(M_3)$. Replace M_1 by M_2 or M_3 in the case when $M = M_1$. Note that a Sylow 3-subgroup of M_2 and M_3 is also a Sylow 3-subgroup of G. Thus by CAYLEY,

$$\Phi(M_2, 3) = \{3_+^{1+4}, S\} \quad \text{and} \quad \Phi(M_3, 3) = \{3^5, S\},$$

and using CAYLEY, we have $N_{M_i}(R) = N(R)$ for each $R \in \Phi(M_i, 3)$. Thus we may suppose $\Phi(M_i, 3) \subseteq \Phi(G, 3)$ for $i = 2, 3$ and each non-trivial element of $\Phi(G, 3)$ is given (up to conjugacy) by (2B). In addition, $|\operatorname{Irr}^0(4S_6)| = 4$, by CAYLEY. Since $|\operatorname{Irr}^0(M_{11})| = 1$ (see [6, p. 18]), it follows that $|\operatorname{Irr}^0(M_3/3^5)| = 2$. If $Q \in \operatorname{Syl}_3(M_{11})$, then $N_{M_{11}}(Q)/Q \simeq SD_{2^4}$ (cf. [3, (2A)]). Since $|N(S)/S| = 2^5$, it follows that $N(S) \simeq S\colon (2 \times SD_{2^4})$ and $|\operatorname{Irr}^0(N(S)/S)| = 14$.

(2C): *The non-trivial radical 2-subgroups R of Co_3 (up to conjugacy) are*

R	$C(R)$	$N/C(R)R$	$\lvert\operatorname{Irr}^0(N/C(R)R)\rvert$
2	$2.S_6(2)$	1	
2^*	$2^* \times M_{12}$	1	
2^2	$2^2 \times S_5$	3	
2^3	$2^3 \times S_3$	F_7^3	
2^4	2^4	A_8	1
$2^2.2^4$	2^2	S_6	1
2_+^{1+6}	2	$L_3(2)$	1
$2^3.2^4$	2^3	$L_3(2)$	1
$2^2.2^6$	2^2	$F_{3^2}^2.S_3$	1
$2.2^3.2^5$	2	S_3	1
$(2.2^3.2^5)^*$	2	S_3	1
$2.2^4.2^4$	2	S_3	1
$(2.2^4.2^4)^*$	2	S_3	1
S	2	1	1,

where $S \in \operatorname{Syl}_2(\mathrm{Co}_3)$, H^* denotes a non-conjugate subgroup of Co_3 which is isomorphic to H, and $F_{n^a}^m$ denotes the Frobenius group with kernel E_{n^a} and complement $\mathbb{Z}_m$.

Proof: If $1 \le i \le 6$, then by [6, p. 134], $G = \mathrm{Co}_3$ has maximal subgroups M_i such that $M_1 \simeq 2.S_6(2)$, $M_2 \simeq 2^4.A_8$, $M_3 \simeq 2 \times M_{12}$, $M_4 \simeq 2^2.[2^7.3^2].S_3$, $M_5 \simeq A_4 \times S_5$ and $M_6 \simeq S_3 \times L_2(8)\colon 3$. Suppose $1 \ne R \in \Phi(G, 2)$ and $W = \Omega_1(Z(R))$, so that $N(R) \le N(W)$. As shown on page 68 of [9] we may suppose $N(R) \le N(A)$ for some $2A$-pure or $2B$-pure elementary abelian 2-subgroup A of G. Indeed, let t and s be the number of involutions of classes $2A$ and $2B$, respectively. Since G has exactly two classes of involutions, it follows that $t + s + 1 = |W|$ is a power of 2. In addition, let X (resp. Y) be the subset of W consisting of involutions of class $2A$ (resp. $2B$) such that the product of any two distinct involutions of X (resp. Y) is an involution of class $2A$ (resp. $2B$). If $X \neq \emptyset$, then X generates a $2A$-pure elementary abelian 2-subgroup A and $N(W) \leq N(A)$. Similarly, if $Y \neq \emptyset$, then Y generates a $2B$-pure elementary abelian 2-subgroup A and $N(W) \leq N(A)$. Suppose $X = Y = \emptyset$. Let $\{x_1, \dots, x_t\}$ and $\{y_1, \dots, y_s\}$ be the subsets of W consisting of involutions of classes $2A$ and $2B$, respectively. Since the product of any two distinct involutions of class $2A$ is in class $2B$ and the product of any two distinct involutions of class $2B$ is in class $2A$, it follows that $x_1x_iy_j$ is either 1 or an involution of class $2A$ for any $i \neq 1$ and $1 \leq j \leq s$, and moreover, $x_1x_iy_j = x_1x_{i'}y_{j'}$ if and only if $i = i'$ and $j = j'$. Thus $t+1 \geq (t-1)s$. Similarly, $s + 1 \geq (s-1)t$. Since $t + s + 1$ is a power of 2, it follows that either $t = 1$ or $s = 1$. In the former case W has a unique involution of class $2A$, so that $N(W) \leq N(2A)$. In the latter case $N(W) \leq N(2B)$. It follows that we may suppose $N(W) \leq N(A)$ for some $2A$-pure or $2B$-pure elementary abelian 2-subgroup A of G

By Lemmas 5.8, 5.9 and 5.10 of [9], $N(A) \leq M_i$ for $1 \leq i \leq 5$, except when A is a $2B$-pure and $A \simeq 2^3$, in which case $N(A) \leq M_6$ (see the remark after Lemma 5.10 of [9]). Thus $N(R) \leq_G M_i$ for some i, and we may suppose $R \in \Phi(M_i, 2)$ satisfying $N(R) = N_{M_i}(R)$.

Using CAYLEY, we can identify the maximal subgroups M_i for $1 \leq i \leq 5$ with the normalizers of 2-subgroups of G, and M_6 with the normalizer of a 3-subgroup. Applying the approach described before (2A) to each maximal subgroup M_i, we can classify the radical 2-subgroups of M_i. For each radical subgroup R, the central series of R which can be calculated by CAYLEY gives the structure of R.

(1) Let $2 = O_2(M_1)$. Using CAYLEY, we have that

$$\Phi(M_1, 2) = \{2, 2^2.2^4, 2_+^{1+6}, 2^2.2^6, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\}, \tag{2.2}$$

where $S \in \mathrm{Syl}_2(G)$. By CAYLEY, $N_{M_1}(R) = N(R)$ for each

$$R \in \Phi(M_1, 2) \backslash \{2^2.2^6\} \quad \text{and} \quad N_{M_1}(2^2.2^6) = 2^2.2^6.F_{3^2}^2.2.$$

(2) $2^4 = O_2(M_2)$. By CAYLEY,

$$\Phi(M_2, 2) = \{2^4, 2_+^{1+6}, 2^3.2^4, 2^2.2^6, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\}, \tag{2.3}$$

$N_{M_2}(R) = N(R)$ for each $R \in \Phi(M_2, 2) \backslash \{2^2.2^6\}$ and $N_{M_2}(2^2.2^6) = 2^2.2^6.S_3^2.S_3$.

(3) Let $2^* = O_2(M_3)$. Then 2^* is generated by a $2B$-element. By [3, (2D)], $\Phi(M_{12}, 2) = \{1, \mathbb{Z}_2, E_4, E_4.E_8, 2_+^{1+4}, Q\}$, where $Q \in \mathrm{Syl}_2(M_{12})$. Thus

$$\Phi(M_3, 2) = \{2^*, 2^2, 2^3, 2 \times 2_+^{1+4}, 2^3.2^3, 2^3.2^3.2\}, \tag{2.4}$$

where $2^2 = 2^* \times \mathbb{Z}_2$, $2^3 = 2^* \times E_4$, $2^3.2^3 = 2^* \times E_4.E_8$, and $2^3.2^3.2 = 2 \times Q$. Moreover, by CAYLEY, $N_{M_3}(R) \neq N(R)$ for $R \in \Phi(M_3, 2) \backslash \{2^*\}$.

(4) Let $2^2.2^6 = O_2(M_4)$. By CAYLEY,

$$\Phi(M_4, 2) = \{2^2.2^6, (2.2^3.2^5)^*, 2.2^4.2^4, (2.2^4.2^4)^*, S\}, \tag{2.5}$$

and moreover, $N_{M_4}(R) = N(R)$ for all $R \in \Phi(M_4, 2)$.

(5) Let $2^2 = O_2(M_5)$. Since $\Phi(S_5, 2) = \{1, \mathbb{Z}_2, E_4, D_8\}$, it follows that

$$\Phi(M_5, 2) = \{2^2, 2^3, (2^4)^*, 2^2 \times D_8\}, \tag{2.6}$$

where $2^3 = 2^2 \times \mathbb{Z}_2$ and $(2^4)^* = 2^2 \times E_4$. Moreover, by CAYLEY $N_{M_5}(R) \neq N(R)$ for each $R \in \Phi(M_5, 2) \backslash \{2^2\}$.

(6) Finally, if $\mathbb{Z}_2 \in \mathrm{Syl}_2(S_3)$ and $Q \in \mathrm{Syl}_2(M_6)$, then by CAYLEY,

$$\Phi(M_6, 2) = \{\mathbb{Z}_2, 2^3, Q\}$$

and moreover, $N_{M_6}(R) \neq N(R)$ unless $R = 2^3$.

Thus the nontrivial radical subgroups are given by (2C), and their normalizers and centralizers are obtained by CAYLEY.

Denote by $D(B)$ a defect group of a block B, $\mathrm{Irr}(B)$ the set of irreducible ordinary characters of B, and $\mathrm{k}(B) = |\,\mathrm{Irr}(B)|$.

(2D): *Let $G = \mathrm{Co}_3$ and let $\mathrm{Blk}^0(G, p)$ be the set of p-blocks with a non-trivial defect group.*

(a) *If $p = 5$ or 3, then $\mathrm{Blk}^0(G, p) = \{B_0, B_1\}$ such that $D(B_1) \simeq \mathbb{Z}_p$. In the notation of* [6, p. 135]

$$\mathrm{Irr}(B_1) = \begin{cases} \{\chi_5, \chi_{12}, \chi_{29}, \chi_{35}, \chi_{39}\} & \text{if } p = 5, \\ \{\chi_{31}, \chi_{32}, \chi_{36}\} & \text{if } p = 3. \end{cases}$$

(b) *If $p = 2$, then $\mathrm{Blk}^0(G, 2) = \{B_0, B_1, B_2\}$ such that $D(B_1) =_G 2$, $D(B_2) =_G 2^3$ and $\ell(B_2) = 5$. In the notation of* [6, p. 135], $\mathrm{Irr}(B_1) = \{\chi_{33}, \chi_{34}\}$ *and*

$$\mathrm{Irr}(B_2) = \{\chi_6, \chi_7, \chi_{18}, \chi_{19}, \chi_{29}, \chi_{32}, \chi_{38}, \chi_{39}\}.$$

Moreover,

$$(2.7) \qquad \ell(B_1) = \begin{cases} 4 & \textit{if } p=5, \\ 2 & \textit{if } p=3, \\ 1 & \textit{if } p=2. \end{cases} \qquad \ell(B_0) = \begin{cases} 18 & \textit{if } p=5, \\ 20 & \textit{if } p=3, \\ 10 & \textit{if } p=2. \end{cases}$$

Proof: If $B \in \mathrm{Blk}(G,p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character, so by (2A), (2B) and (2C), $D \in_G \{5,3,2,2^3\}$. Moreover, for each such D, $|\,\mathrm{Irr}^0(C(D)D/D)| = 1$, so G has exactly one block with a defect group D.

Using the method of central characters on elements of classes 2B, 3C and 5A, we have $\mathrm{Irr}(B)$ given as in (2D). If $D(B)$ is cyclic, then $\ell(B)$ is the number of B-weights. Thus $\ell(B_1)$ is given by (2.7).

Suppose $p = 2$ and $B = B_2$. Since B_2 is non-principal, it follows that $D(B_2) =_G 2$ or 2^3. If $D(B_2) =_G 2$, then $\mathrm{k}(B_2) \leq 2$ (cf. [8, p. 170]). Since $\mathrm{k}(B) = 8$, it follows that $D(B) =_G 2^3$. Let $K = C(2^*) = 2^* \times M_{12}$, and let $b \in \mathrm{Blk}(K)$ such that $b^G = B$. Then $D(b) \simeq 2^3$ and $b = b_0 \times b_1$, where b_0 is the principal block of 2^* and $b_1 \in \mathrm{Blk}(M_{12})$ with $D(b_1) \simeq E_4$. By [10, Theorem 8.2], M_{12} has exactly one block b_1 with a defect group E_4 and $\ell(b_1) = 3$. Thus K has exactly one block b such that $b^G = B$, and moreover, $\ell(b) = 3$. As shown on page 72 of [9], 2^3 is $2B$-pure, so that $\mathrm{k}(B) = \ell(B) + \ell(b)$ and $\ell(B) = 5$.

Finally, let $\ell(G)$ be the number of p-regular conjugacy G-classes. Then $\ell(B_0)$ can be calculated by the following equation due to Brauer,

$$\ell(G) = \bigcup_{B \in \mathrm{Blk}^0(G,p)} \ell(B) + |\,\mathrm{Irr}^0(G)|.$$

This completes the proof.

(2E): *Let $G = \mathrm{Co}_3$ and B a p-block of G with a non-cyclic defect group. Then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof: If $B = B_0$, then (2E) follows by (2.1) and (2A)–(2D). Suppose $p = 2$ and $B = B_2$. Let $R = D(B_2) = 2^3$, and let $\theta = \theta_1 \times \theta_2$ be a character of $C_G(R) = 2^3 \times S_3$, where θ_1 is the trivial character of R and $\theta_2 \in \mathrm{Irr}(S_3)$ is the character of degree 2. Since θ is uniquely determined in $\mathrm{Irr}(C_G(R))$ by its degree, it follows that $N(\theta) = N(R)$. By (2C), $N(R) = (2^3 \times S_3).F_7^3$. Let $F_7^3 = \langle\sigma\rangle \rtimes \langle\tau\rangle$, where $|\sigma| = 7$ and $|\tau| = 3$. Since σ stabilizes θ, it follows by Clifford theory that θ has 7 extemsions to $H = (2^3 \times S_3).\langle\sigma\rangle$. Since τ stabilizes θ and τ normalizes $\langle\sigma\rangle$, it follows that τ permutes these extensions. Since τ has order 3 (modulo H), it follows that each $\langle\tau\rangle$-orbit on the extensions contains either one or three characters. But θ has only 7 extensions, so at least one $\langle\tau\rangle$-orbit contains only one character θ'. Thus θ' has an extension to $N(R) = \langle H, \tau\rangle$, so that θ has an extension to $N(R)$. By [3, (1.4)], $\mathcal{W}(B_2) = |\operatorname{Irr}^0(N(\theta)/C(R)R)| = |\operatorname{Irr}(F_7^3)| = 5$, since F_7^3 has 3 linear characters and 2 irreducible characters of degree 3 (cf. [5, Proposition 3]).

3. Radical chains

The notation and terminology of Sections 1 and 2 are continued in this section. Let $G = \mathrm{Co}_3$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$.

(3A): *Follow the notation of* (2A). *The radical* 5*-chains* C *of* G (*up to conjugacy*) *are*

C	$N(C)$	C	$N(C)$
$C(1): 1$	G	$C(2): 1 < 5$	$(5 \times A_5).4$
$C(3): 1 < 5 < 5^2$	$(5 \times F_5^2).4$	$C(4): 1 < 5_+^{1+2}$	$5_+^{1+2}.24.2$,

where $5^2 \in \operatorname{Syl}_5(5 \times A_5)$.

Proof: Suppose C is a radical 5-chain given by (1.1) with $|C| \geq 1$, so that $P_0 = 1$. By definition, P_1 is a radical subgroup of G, and so by (2A), we may suppose $P_1 = 5$ or 5_+^{1+2}. If $P_1 = 5_+^{1+2}$, then $|C| = 1$ and $C =_G C(4)$. If $P_1 = 5$, then $N(C_1) = N(P_1) = (5 \times A_5).4$, so that either $C =_G C(2)$ or $|C| \geq 2$. If $|C| \geq 2$, then by definition, P_2 is a radical 5-subgroup of $N(P_1)$ and $P_2 \neq P_1$, so that $P_2 = 5^2 \in \operatorname{Syl}_5(N(P_1))$, $|C| = 2$ and $C =_G C(3)$.

Suppose $C = C(3)$. Then $N(C) = N_{N(5)}(5^2) = (5 \times N_{A_5}(X)).4$, where X is a Sylow 5-subgroup of A_5. Thus $N(C) \simeq (5 \times F_5^2).4$, since $|N_{A_5}(X)| = 10$ and $N_{A_5}(X)$ contains 2 linear characters and 2 irreducible characters of degree 2 (cf. [5, Proposition 3]).

In the notation of (2B), define the radical 3-chains $C(i)$ for $1 \leq i \leq 6$ as follows:

$$
\begin{array}{lll}
 & C(1): 1 & C(2): 1 < 3 \\
 & C(3): 1 < 3 < S' & C(4): 1 < 3_+^{1+4} \\
(3.1) & C(5): 1 < 3^5 < S & C(6): 1 < 3^5,
\end{array}
$$

where $S' \in \operatorname{Syl}_3(S_3 \times L_2(8){:}\, 3)$.

(3B):

(a) *Let* $\mathcal{R}^0(G)$ *be the* G*-invariant subfamily of* $\mathcal{R}(G)$ *such that* $\mathcal{R}^0(G)/G = \{C(i) : 1 \leq i \leq 6\}$. *Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B_0, d)$$

for all integers $d \geq 0$.

(b) *Suppose C is a chain given by (3.1). Then*

C	$N(C)$	C	$N(C)$
$C(1)$	G	$C(2)$	$S_3 \times L_2(8) : 3$
$C(3)$	$S'.2^2$	$C(4)$	$3_+^{1+4} : 4S_6$
$C(5)$	$S : (2 \times SD_{2^4})$	$C(6)$	$3^5 : (2 \times M_{11})$.

Proof: Let $C' : 1 < S$ and $C'' : 1 < 3_+^{1+4} < S$. By CAYLEY,

$$N(C') = N(C'') = N(S).$$

If $\psi \in \operatorname{Irr}(N(C'), B, d)$ for some block B and integer d, then by (1.3), $\psi \in \operatorname{Irr}(N(C'))$, $B(\psi)^G = B$ and $\mathrm{d}(\psi) = d$. Since $N(C') = N(C'')$, it follows that $\psi \in \operatorname{Irr}(N(C''))$, the block $B(\psi)$ is also a block of $N(C'')$ and $\mathrm{d}(\psi) = d$. By (1.3), $\psi \in \operatorname{Irr}(N(C''), B, d)$. Conversely, $\operatorname{Irr}(N(C''), B, d) \subseteq \operatorname{Irr}(N(C'), B, d)$, so that

$$\operatorname{Irr}(N(C'), B, d) = \operatorname{Irr}(N(C''), B, d).$$

Thus

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(C''), B, d)$$

and

$$(-1)^{|C'|}\mathrm{k}(N(C'), B, d) + (-1)^{|C''|}\mathrm{k}(N(C''), B, d) = 0,$$

so that we can delete C' and C'' in the sum (1.4).

Suppose C is a radical 3-chain given by (1.1). Then P_1 is radical in G and we may suppose $P_1 \in \Phi(G, 3)$. If $P_1 = 3$, then by (2B),

$$N(C_1) = N(P_1) = S_3 \times L_2(8){:}\ 3$$

and moreover, $\Phi(N(C_1), 3) = \{3, S'\}$. Thus either $C =_G C(2)$ or $|C| \geq 2$. In the latter case we may suppose $P_2 \in \Phi(N(P_1), 3)$, since P_2 is radical in $N(P_1)$, Thus $P_2 =_G S'$ and $N(C_2) = N_{N(P_1)}(S')$. Since S' is also a Sylow 3-subgroup of $N(C_2)$, it follows that $\Phi(N(C_2), 3) = \{S'\}$ and so $C = C_2 =_G C(3)$. By CAYLEY,

$$N(C(3)) = S'.2^2.$$

Suppose $P_1 = 3_+^{1+4}$. As shown in the proof of (2B) we have

$$\Phi(N(P_1), 3) = \{3_+^{1+4}, S\}, \quad \text{where } S \in \mathrm{Syl}_3(G).$$

Either $C =_G C(4)$ or $|C| \geq 2$. In the latter case $P_2 =_G S$ and $C =_G C''$. Similarly, if $P_1 = 3^5$, then either $C =_G C(6)$ or $C =_G C(5)$. Finally, if $P_1 =_G S$, then $C =_G C'$. This proves (3B) (a). The proof of (3B) (b) follows from that of (2B).

Following the notation of (2C), we define radical 2-chains: $C(i)$ for $1 \leq i \leq 16$ as follows:

$$\begin{array}{lll} & C(1): 1 & C(2): 1 < 2 \\ & C(3): 1 < 2 < 2^2.2^6 & C(4): 1 < 2^2.2^6 \\ & C(5): 1 < 2^4 < 2_+^{1+6} & C(6): 1 < 2^4 \\ & C(7): 1 < 2^4 < 2^2.2^6 & C(8): 1 < 2^4 < 2^2.2^6 < (2.2^3.2^5)^* \\ & C(9): 1 < 2^* < 2^3.2^3 & C(10): 1 < 2^* \\ & C(11): 1 < 2^* < 2 \times 2_+^{1+4} & C(12): 1 < 2^* < 2^3 < (2^4)^* \\ & C(13): 1 < 2^* < 2^3 & C(14): 1 < 2^* < 2^3.2^3 < 2^3.2^3.2 \\ (3.2) & C(15): 1 < 2^3 < (2^4)^* & C(16): 1 < 2^3, \end{array}$$

where $2^3.2^3, 2 \times 2_+^{1+4}, 2^3.2^3.2 \in \Phi(M_3, 2)$, and $(2^4)^* \in \mathrm{Syl}_2(N(2^3))$. We have the following proposition:

(3C):

(a) *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that $\mathcal{R}^0(G)/G = \{C(k)\colon k = 1, 2, \cdots, 16\}$. Then*

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d)$$

for all integers $d \geq 0$ and for each block B with a non-cyclic defect group.

(b) *Let C be a chain given by* (3.2). *Then*

C	$N(C)$	C	$N(C)$
$C(1)$	Co_3	$C(2)$	$2.S_6(2)$
$C(3)$	$2^2.2^6.F_{3^2}^2.2$	$C(4)$	$2^2.2^6.F_{3^2}^2.S_3$
$C(5)$	$2_+^{1+6}.L_3(2)$	$C(6)$	$2^4.A_8$
$C(7)$	$2^2.2^6.S_3.S_3$	$C(8)$	$(2.2^3.2^5)^*.S_3$
$C(9)$	$2^3.2^3.S_3$	$C(10)$	$2^* \times M_{12}$
$C(11)$	$2 \times 2_+^{1+4}.S_3$	$C(12)$	$2^2 \times A_4$
$C(13)$	$2 \times A_4 \times S_3$	$C(14)$	$2^3.2^3.2$
$C(15)$	$(2^4)^*.F_7^3$	$C(16)$	$(2^3 \times S_3).F_7^3$

Proof: (a) Suppose C' is a radical 2-chain such that

$$C': 1 < P_1' < \cdots < P_m'. \tag{3.3}$$

Let $C \in \mathcal{R}(G)$ be given by (1.1) with $P_1 \in \Phi(G, 2)$.

CASE (1): Let $R \in \Phi(M_1, 2)\backslash\{2, 2^2.2^6\}$, so that by the proof (2C) (1), $R \in \Phi(G, 2)$. Let $\mathcal{X}(R)$ and $\mathcal{Y}(R)$ be subsets of radical chains such that

$$\begin{aligned} \mathcal{X}(R) &= \{C' \in \mathcal{R}/G : P_1' = R\}, \quad \text{and} \\ \mathcal{Y}(R) &= \{C' \in \mathcal{R}/G : P_1' = 2, P_2' = R\}. \end{aligned} \tag{3.4}$$

By the proof (2C) (1), any two distict subgroups of $\Phi(M_1, 2)\backslash\{2, 2^2.2^6\}$ are not conjugate in G, so that any two distinct chains of $\mathcal{X}$ or $\mathcal{Y}$ are not G-conjugate. Define $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ be subsets of $\mathcal{R}$ consisting of all G-conjugates of chains in $\mathcal{X}(R)$ and $\mathcal{Y}(R)$, respectively. Then $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ are G-invariant with $\mathcal{X}(R)$ and $\mathcal{Y}(R)$ as their representative sets for G-orbits. For $C' \in \mathcal{X}(R)$ given by (3.3) with $P_1' = R$, define

$$g(C')\colon 1 < 2 < P_1' = R < P_2' < \cdots < P_m'. \tag{3.5}$$

Then $g(C') \in \mathcal{M}^0(R)$ and we may suppose $g(C') \in \mathcal{Y}(R)$. As shown in the proof of (2C) (1) $N(R) = N(1 < 2 < R)$, so that by (1.2), $N(C') = N(g(C'))$. Conversely, if $g(C') : 1 < P_1' = 2 < P_2' = R < P_3' < \cdots < P_m' \in \mathcal{Y}(R)$, then $C' : 1 < P_2' = R < P_3' < \cdots < P_m'$ is a chian of $\mathcal{X}(R)$. So the map g from $\mathcal{X}(R)$ to $\mathcal{Y}(R)$ is onto. Since $\Phi(M_1, 2)\backslash\{2, 2^2.2^6\}$ is a subset of $\Phi(G, 2)$, the map g is a bijection. Extend the map g to a bijection, which is also denoted by g between $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$. Since $N(C') = N(g(C'))$, it follows by (1.3) that for any block B and for any integer $d \geq 0$, $\mathrm{Irr}(N(C'), B, d) = \mathrm{Irr}(N(g(C')), B, d)$, so that

$$\mathrm{k}(N(C'), B, d) = \mathrm{k}(N(g(C')), B, d). \tag{3.6}$$

Since $(-1)^{|C'|}\,\mathrm{k}(N(C'), B, d) + (-1)^{|g(C')|}\,\mathrm{k}(N(g(C')), B, d) = 0$ for $C' \in \mathcal{X}(R)$, we can delete the chains of $\mathcal{M}^+(R)$ and $\mathcal{M}^0(R)$ in the right hand side of (1.4). Thus we may suppose

$$C \notin \bigcup_{R \in \Phi(M_1,2)\backslash\{2,2^2.2^6\}} (\mathcal{M}^+(R) \cup \mathcal{M}^0(R)).$$

In particular,

$$P_1 \notin \Phi(M_1, 2)\backslash\{2, 2^2.2^6\} = \{2^2.2^4, 2_+^{1+6}, 2.2^3.2^5, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\},$$

and if $P_1 = 2$, then $C =_G C(2)$ or $P_2 = 2^2.2^6$.

CASE (2): Replace R by $2.2^4.2^4 \in \Phi(M_4, 2)$ and 2 by $2^2.2^6 \in \Phi(M_4, 2)$ in the definition (3.4), and repeat the proof above. Then we may suppose $P_1 \neq 2.2^4.2^4$, and if $P_1 = 2^2.2^6$, then $P_2 \neq_G 2.2^4.2^4$. By CAYLEY, $N_{M_1}(2^2.2^6) = N(C(3)) \simeq 2^2.2^6.F_{3^2}^2.2$, and $N(C(3))$ has 4 radical 2-subgroups. A Sylow 2-subgroup Q of $M_4/2^2.2^6 \simeq F_{3^2}^2.S_3$ is elementary abelian of order 4 and by (2.5), the preimage in M_4 of a subgroup of Q is a radical subgroup of M_4. In particular, each radical subgroup of $N(C(3))$ is a radical subgroup of M_4. Using CAYLEY, we have that

$$\Phi(2^2.2^6.F_{3^2}^2.2, 2) = \{2^2.2^6, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\} \subseteq \Phi(M_4, 2)$$

such that $N_{N(C(3))}(R) = N(R) = N_{M_4}(R)$ for $R \in \Phi(2^2.2^6.F_{3^2}^2.2, 2)\backslash\{2^2.2^6\}$.

Let $\Omega = \{(2.2^3.2^5)^*, (2.2^4.2^4)^*, S\} \subseteq \Phi(2^2.2^6.F_{3^2}^2.2, 2)$. Given $Q \in \Omega$, similar to Case (1) we define G-invariant subfamilies $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ of $\mathcal{R}(G)$, such that the representatives for G-orbits of $\mathcal{L}^+(Q)$ and $\mathcal{L}^0(Q)$ are given as follows:

$$\begin{aligned} &\mathcal{L}^+(Q)/G = \{C' \in \mathcal{R}/G\colon P_1' = 2^2.2^6,\ P_2' = Q\}, \quad \text{and} \\ &\mathcal{L}^0(Q)/G = \{C' \in \mathcal{R}/G\colon P_1' = 2,\ P_2' = 2^2.2^6,\ P_3' = Q\}. \end{aligned} \tag{3.7}$$

If $C' \in \mathcal{L}^+(Q)/G$ is given by (3.3) with $P_1' = 2^2.2^6$ and $P_2' = Q$, then

$$g(C')\colon 1 < 2 < P_1' = 2^2.2^6 < P_2' = Q < P_3' < \cdots < P_m'$$

is a chain of $\mathcal{L}^0(Q)$ and moreover, g induces a bijection between $\mathcal{L}^+(Q)/G$ and $\mathcal{L}^0(Q)/G$ (see the proof of Case (1)). Since $N(2^2.2^6) = M_4$ and $N_{M_4}(Q) = N(1 < 2 < 2^2.2^6 < Q) = N(Q)$, it follows by (1.2) that $N(C') = N(g(C'))$. Thus (3.6) holds for each $C' \in \mathcal{L}^+(Q)/G$. A proof similar to that of Case (1) shows that we may suppose

$$C \notin \bigcup_{Q \in \Omega} (\mathcal{L}^+(Q) \cup \mathcal{L}^0(Q)). \tag{3.8}$$

In particular, if $P_1 = 2^2.2^6$, then by (2.5), $C =_G C(4)$ and moreover, if $P_1 = 2$ and $P_2 =_G 2^2.2^6$, then $C =_G C(3)$.

By Case (1), we may suppose $P_1 \notin \Phi(M_1, 2)\backslash\{2, 2^2.2^6\}$ and by Case (2) above we may suppose $P_1 \neq 2.2^4.2^4$. So by (2C), we may suppose

$$P_1 \in \Phi(G, 2)\backslash(\{2.2^4.2^4\} \cup \Phi(M_1, 2)\backslash\{2, 2^2.2^6\}) = \{2, 2^*, 2^2, 2^3, 2^4, 2^3.2^4, 2^2.2^6\},$$

and moreover, if $P_1 \in \{2, 2^2.2^6\}$, then by Cases (1) and (2),

$$C \in_G \{C(2), C(3), C(4)\}.$$

CASE (3): Let $\mathcal{M}^+(2^3.2^4)$ and $\mathcal{M}^0(2^3.2^4)$ be defined as in Case (1) with R replaced by $2^3.2^4$ and 2 by 2^4. A proof similar to that of Case (1) shows that (3.6) holds for $C' \in \mathcal{M}^+(2^3.2^4)$, so that we may suppose $P_1 \neq 2^3.2^4$ and if $P_1 = 2^4$, then $P_2 \neq_G 2^3.2^4$. Let $\{2.2^3.2^5, (2.2^3.2^5)^*, S\}$ be a subset of $\Phi(M_2, 2)$ given by (2.3) and let $R \in \{2.2^3.2^5, (2.2^3.2^5)^*, S\}$. Using CAYLEY, we can check that $N(R) = N_{M_2}(R) \leq N(2_+^{1+6}) = 2_+^{1+6}.L_3(2)$, so that $R \in_G \Phi(N(2_+^{1+6}), 2)$. Applying the approach described before (2B) to the subgroup $N(2_+^{1+6})$, we have that $N(2_+^{1+6})$ has exactly 4 radical 2-subgroups, so that

$$\Phi(2_+^{1+6}.L_3(2), 2) = \{2_+^{1+6}, 2.2^3.2^5, (2.2^3.2^5)^*, S\} \subseteq \Phi(M_2, 2),$$

and moreover, $N_{N(2_+^{1+6})}(R) = N_{M_2}(R) = N(R)$ for all $R \in \Phi(N(2_+^{1+6}), 2)$. Let $\Omega' = \{2.2^3.2^5, (2.2^3.2^5)^*, S\} \subseteq \Phi(2_+^{1+6}.L_3(2), 2)$, and $W \in \Omega'$. Similar to the proof of Case (1) we define G-invariant subfamilies $\mathcal{K}^+(W)$ and $\mathcal{K}^0(W)$ of $\mathcal{R}(G)$, such that

$$\text{(3.9)} \qquad \begin{aligned} \mathcal{K}^+(W)/G &= \{C' \in \mathcal{R}/G\colon P_1' = 2^4,\ P_2' = W\}, \quad \text{and} \\ \mathcal{K}^0(W)/G &= \{C' \in \mathcal{R}/G\colon P_1' = 2^4,\ P_2' = 2_+^{1+6},\ P_3' = W\}. \end{aligned}$$

A similar proof to that of Case (1) shows that there is a bijection g between $\mathcal{K}^+(W)$ and $\mathcal{K}^0(W)$ such that $N(C') = N(g(C'))$ for each $C' \in \mathcal{K}^+(W)/G$, so that (3.6) holds and we may suppose

$$C \notin \bigcup_{W \in \Omega'} (\mathcal{K}^+(W) \cup \mathcal{K}^0(W)).$$

Thus if $P_1 = 2^4$, then by (2.3), we may suppose $P_2 \in \{2_+^{1+6}, 2^2.2^6, (2.2^4.2^4)^*\}$, and moreover, if $P_2 =_G 2_+^{1+6}$, then $C =_G C(5)$.

By CAYLEY, $N_{M_2}(2^2.2^6) \simeq 2^2.2^6.S_3.S_3$, and

$$\Phi(2^2.2^6.S_3.S_3, 2) = \{2^2.2^6, (2.2^3.2^5)^*, (2.2^4.2^4)^*, S\},$$

and moreover,

$$N_{N_{M_2}(2^2.2^6)}(R) = N_{M_2}(R) = N(R)$$

for each $R \in \Phi(N_{M_2}(2^2.2^6), 2)\backslash\{2^2.2^6\}$. Replace W by $(2.2^4.2^4)^*$ and 2_+^{1+6} by $2^2.2^6$ in the definition (3.9). We may suppose $P_2 \neq_G (2.2^4.2^4)^*$, and if $P_1 = 2^4$ and $P_2 =_G 2^2.2^6$, then $P_3 \not\simeq 2.2^4.2^4$.

Let C' be the chain $1 < 2^4 < 2^2.2^6 < (2.2^3.2^5)^* < S$, and $g(C')$ be the chain $1 < 2^4 < 2^2.2^6 < S$. Then $N(C') = N(g(C')) = S$ and so by (1.3),

$\mathrm{Irr}(N(C'), B, d) = \mathrm{Irr}(N(g(C')), B, d)$ for any block B and integer d, so that (3.6) holds and we can delete the chains C' and $g(C')$ in the sum (1.4). If $P_1 = 2^4$ and $|C| \geq 2$, then $P_2 \in \{2_+^{1+6}, 2^2.2^6\}$, since $P_2 \neq_G (2.2^4.2^4)^*$. Moreover, if $P_2 =_G 2_+^{1+6}$, then $C =_G C(5)$ as shown above; if $P_2 =_G 2^2.2^6$ and $|C| = 2$, then $C =_G C(7)$. If $P_2 =_G 2^2.2^6$ and $|C| \geq 3$, then $P_3 \neq_G (2.2^4.2^4)^*$ and so we may suppose $P_3 \in \{(2.2^3.2^5)^*, S\}$. Since C is not conjugate to the chain $1 < 2^4 < 2^2.2^6 < S$, it follows that $P_3 =_G (2.2^3.2^5)^*$ and $C =_G C(8)$ whenever $|C| = 3$. Suppose moreover, $P_3 =_G (2.2^3.2^5)^*$ and $|C| \geq 4$. Then $P_4 =_G S$, since by the order, S and $(2.2^3.2^5)^*$ are the only two radical 2-subgroups (up to conjugacy) of $N((2.2^3.2^5)^*)$. In this case C is G-conjugate to $1 < 2^4 < 2^2.2^6 < (2.2^3.2^5)^* < S$, which is impossible. It follows that if $P_1 = 2^4$, then $C \in_G \{C(5), C(6), C(7), C(8)\}$.

CASE (4): If $P_1 = 2^3$, then $C = C(15)$ or $C(16)$, since $N(2^3) \simeq (2^3 \times S_3).F_7^3$ and $\Phi(N(2^3), 2) = \{2^3, (2^4)^*\}$, where $(2^4)^*$ is a Sylow 2-subgroup of $N(2^3)$.

CASE (5): Let C': $1 < 2^* < S'$ and $g(C')$: $1 < 2^* < 2 \times 2_+^{1+4} < S'$, where $2 \times 2_+^{1+4}, S' = 2^3.2^3.2 \in \Phi(M_3, 2)$. Then $N(C') = N(g(C')) = S'$ and by (1.3), $\mathrm{Irr}(N(C'), B, d) = \mathrm{Irr}(N(g(C')), B, d)$ for any block B and integer d. So (3.6) holds and we may delete the chains C' and $g(C')$ in the sum (1.4).

By [3, (2D)], $N_{M_3}(2^2) = 2 \times N_{M_{12}}(\mathbb{Z}_2) \simeq 2^2 \times S_5$ and by CAYLEY, $O_2(N_{M_3}(2^2))$ is conjugate to $O_2(M_5) = 2^2$, so we may identify $O_2(N_{M_3}(2^2))$ with $O_2(M_5)$. As shown in the proof of (2C) (5), $\Phi(S_5, 2) = \{1, \mathbb{Z}_2, E_4, D_8\}$ and $\Phi(M_5, 2) = \{2^2, 2^3, (2^4)^*, 2^2 \times D_8\}$, where $2^3 = 2^2 \times \mathbb{Z}_2$ and $(2^4)^* = 2^2 \times E_4$. Since $N_{M_3}(2^2) = 2^2 \times S_5 \leq M_5 = A_4 \times S_5$, it follows that

$$\Phi(2^2 \times S_5, 2) = \{2^2, 2^3, (2^4)^*, 2^2 \times D_8\} = \Phi(M_5, 2).$$

Let $\Omega^* = \{2^3, (2^4)^*, 2^2 \times D_8\} \subseteq \Phi(N_{M_3}(2^2), 2)$. For each $R \in \Omega^*$, $R = 2^2 \times R'$ for some $R' \leq S_5$, $N_{N_{M_3}(2^2)}(R) \simeq 2^2 \times N_{S_5}(R')$ and $N_{M_5}(R) \simeq A_4 \times N_{S_5}(R')$. Replace Q by a group Q' in Ω^*, $2^2.2^6$ by 2^2 and 2 by 2^* in the definition (3.7). If $C' \in \mathcal{L}^+(Q')$ and P is the final subgroup of C', then $P = 2^2 \times P'$ and $N(C') = 2^2 \times N'$, where P' and N' are some subgroups of S_5. Let $g(C')$ be a chain of $\mathcal{L}^0(Q')$ defined similar to the one after (3.7) (with Q replaced by Q', $2^2.2^6$ by 2^2 and 2 by 2^*). Then P is also the final subgroup of $g(C')$ and $N(g(C')) \simeq A_4 \times N'$. Note that both 2^2 and A_4 have exactly 4 irreducible characters of defect 2, and all of them lie in their principal block. So there is a defect-preserved bijection φ between $\mathrm{Irr}(2^2) = \mathrm{Irr}(B_0(2^2))$ and $\mathrm{Irr}(A_4) = \mathrm{Irr}(B_0(A_4))$.

If $\psi \in \mathrm{Irr}(N(C'), B, d)$, then $\psi = \psi_1 \times \psi_2$, $B(\psi)^G = B$ and the defect $\mathrm{d}(\psi)$ of ψ is d, where $\psi_1 \in \mathrm{Irr}(2^2)$, $\psi_2 \in \mathrm{Irr}(N')$ and $B(\psi)$ is the block of $N(C')$ containing

ψ. Since $N(C') = 2^2 \times N'$ and $\psi \in \mathrm{Irr}(B(\psi))$, it follows that $B(\psi) = B_0(2^2) \times b_1$, where b_1 is the block of N' containing ψ_2. If $\varphi^*(\psi) = \varphi(\psi_1) \times \psi_2$, then $\varphi^*(\psi)$ is an irreducible character of $N(g(C')) = A_4 \times N'$ and moreover, $\varphi^*(\psi)$ is a character of $B_0(A_4) \times b_1$, so that $B(\varphi^*(\psi)) = B_0(A_4) \times b_1$. Since $B_0(2^2)^{A_4} = B_0(A_4)$, it follows that $B(\psi)^{N(g(C'))} = B(\varphi^*(\psi))$ and so $B(\varphi^*(\psi))^G = B$ (see [8, Lemma III.9.2]). Since φ is a defect-preserved bijection, it follows by the definition [7, (5.5)] that φ^* is a defect-preserved injection and $\varphi^*(\psi) \in \mathrm{Irr}(N(g(C')), B, d)$.

Conversely, suppose $\chi \in \mathrm{Irr}(N(g(C')), B, d)$. Since $N(g(C')) = A_4 \times N'$, it follows that $\chi = \chi_1 \times \chi_2$ for some $\chi_1 \in \mathrm{Irr}(A_4)$ and $\chi_2 \in \mathrm{Irr}(N')$. In addition, $B(\chi)^G = B$ and $\mathrm{d}(\chi) = d$. Thus $B(\chi) = B_0(A_4) \times b_2$, where b_2 is the block of N' containing χ_2. Since $\mathrm{Irr}(B_0(A_4)) = \mathrm{Irr}(A_4)$ and φ is a bijection, there is a character $\psi_1 \in \mathrm{Irr}(2^2)$ such that $\varphi(\psi_1) = \chi_1$. If $\psi = \psi_1 \times \chi_2$, then $\mathrm{d}(\psi) = \mathrm{d}(\chi) = d$ and $\psi \in \mathrm{Irr}(B_0(2^2) \times b_2)$, so that $B(\psi) = B_0(2^2) \times b_2$. Since $B(\psi)^{N(g(C'))} = B_0(A_4) \times b_2 = B(\chi)$ and since $B(\chi)^G = B$, it follows that $B(\psi)^G = B$ and then $\psi \in \mathrm{Irr}(N(C'), B, d)$. Thus $\varphi^*(\psi) = \chi$ and φ^* is a bijection. It follows that (3.6) holds and we may suppose (3.8) holds for $Q' \in \Omega^*$. Similarly, if $C' : 1 < 2^* < 2^2$ and $g(C') : 1 < 2^2$, then $N(g(C')) \simeq A_4 \times S_5$ and $N(C') \simeq 2^2 \times S_5$. Thus (3.6) still holds and we may delete them in the sum (1.4).

Suppose $P_1 = 2^*$. If $|C| = 1$, then $C =_G C(10)$ and we may suppose $|C| \geq 2$. By (2.4) and the proof in Case (5), we may suppose $P_2 \in \{2^3, 2 \times 2_+^{1+4}, 2^3.2^3\}$ and moreover, if $P_2 =_G 2 \times 2_+^{1+4}$, then $|C| = 2$ and $C =_G C(11)$, since

$$\Phi(N_{M_3}(2 \times 2_+^{1+4}), 2) = \{2 \times 2_+^{1+4}, S'\}$$

and $1 < 2^* < 2 \times 2_+^{1+4} < S'$ is deleted from the sum (1.4). If $P_2 =_G 2^3$, then $C =_G C(13)$ or $C(12)$ according as $|C| = 2$ or 3, since $\Phi(N_{M_3}(2^3), 2) = \{2^3, (2^4)^*\}$ (cf. Case (4)). If $P_2 =_G 2^3.2^3$, then $C =_G C(9)$ or $C(14)$ according as $|C| = 2$ or 3, since by the order, $\Phi(N_{M_3}(2^3.2^3), 2) = \{2^3.2^3, S'\}$. Hence if $P_1 \in \{2^*, 2^2\}$, then $C \in_G \{C(i)\colon i = 9, \ldots, 14\}$.

(b) The proof of (b) follows easily by that of (a) above or (2C) (cf. [3, (2D)]).

4. The proof of Dade's conjecture for Co_3

The notation and terminology of Sections 2 and 3 are continued in this section. The character table of the normalizer $N(C)$ of each radical p-chain C given in (3A), (3.1) or (3.2) can be obtained by CAYLEY.

(4A): *Let B be a p-block of the simple Conway group $G = \mathrm{Co}_3$ with a positive defect. If p is odd, then B satisfies the ordinary conjecture of Dade.*

Proof: We may suppose $p = 5$ or 3, and $B = B_0$.

Suppose $p = 5$ and let $C = C(2)$, $C' = C(3)$. By (3A), $N(C) \simeq (5 \times A_5).4$ and $N(C') \simeq (5 \times F_5^2).4$. By CAYLEY, $N(C)$ has 25 irreducible characters, 4 linear characters, 8 characters of degree 3, 5 of degree 4, 4 of degree 5, 2 of degree 12, 1 of degree 16 and degree 20. By Lemma 6.9 of Dade [7], if C'' is a radical p-chain of G such that $N(C'')$ has a p-block b inducing the block B of G, then the final subgroup of C'' is conjugate to a subgroup of $D(B)$. In particular, if $B = B_1$ is the 5-block given by (2D) with $D(B_1) \simeq 5$, then $\mathrm{k}(N(C''), B_1, d) \neq 0$ for some d implies that $C'' =_G C(1)$ or $C(2)$, which are defined in (3A). By Theorem 9.1 of [7], Dade's conjecture holds for the cyclic block B_1, so that by (1.4), $\mathrm{k}(N(C(1)), B_1, d) = \mathrm{k}(N(C), B_1, d)$ for all integers d. By (2D) (a), $\mathrm{k}(G, B_1, d) = 5$ or 0 according as $d = 1$ or $d \neq 1$, so that $\mathrm{Irr}(N(C), B_1, 1)$ contains 5 characters of defect 1 in $\mathrm{Irr}(N(C))$. But $\mathrm{Irr}(N(C))$ has exactly 5 such characters with degrees 5 and 20, so the remaining 20 characters of $\mathrm{Irr}(N(C))$ (with height 0) are characters of $\mathrm{Irr}(B_0(N(C)))$ as G has only two blocks with positive defect. Moreover, by the definition (5.5) of [7], $\mathrm{k}(N(C), B_0, d) = 20$ or 0 according as $d = 2$ or $d \neq 2$. Since $C_G(5^2) = C_{N(C)}(5^2) = 5^2 \trianglelefteq N(C')$, it follows by [8, Corollary V.3.11] that $N(C')$ has only the principal block, so that $\mathrm{Irr}(N(C')) = \mathrm{Irr}(B_0(N(C')))$. By CAYLEY, $N(C')$ has 20 irreducible characters, 8 linear characters, 8 characters of degree 2, 2 of degree 4 and degree 6. By definition, $\mathrm{k}(N(C'), B_0, d) = 20$ or 0 according as $d = 2$ or $d \neq 2$. Thus

$$\mathrm{k}(N(C), B_0, d) = \mathrm{k}(N(C'), B_0, d).$$

for all integers $d \geq 0$. By CAYLEY, the degrees of irreducible characters of $N(C(4)) \simeq 5_+^{1+2}.24.2$ is given by Table 1.

Table 1. The degrees of characters of $\mathrm{Irr}(5_+^{1+2}.24.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	1	1	2	2	2	2	2	2
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}		
2	2	2	2	20	20	20	20	24	24	40	40		

Since $C_G(5_+^{1+2}) = 5 \leq 5_+^{1+2} \trianglelefteq N(C(4))$, it follows that $B_0(N(C(4)))$ is the only block of $N(C(4))$, so that $\mathrm{Irr}(N(C(4))) = \mathrm{Irr}(B_0(N(C(4))))$. By Table 1 and the definition (5.5) of [7],

$$\mathrm{Irr}(N(C(4)), B_0, 2) = \{\xi_{19}, \xi_{20}, \xi_{21}, \xi_{22}, \xi_{25}, \xi_{26}\}$$

and $\mathrm{Irr}(N(C(4)), B_0, 3) = \mathrm{Irr}(N(C(4))) \setminus \mathrm{Irr}(N(C(4)), B_0, 2)$. By (2D) (a),

$$\mathrm{Irr}(B_0(G)) = \mathrm{Irr}(G) \setminus (\mathrm{Irr}(B_1) \cup \mathrm{Irr}^0(G))$$

and so by [6, p. 135], $\operatorname{Irr}(G, B_0, 2) = \{\chi_{10}, \chi_{11}, \chi_{15}, \chi_{27}, \chi_{30}, \chi_{40}\}$ and

$$\operatorname{Irr}(G, B_0, 3) = \{\chi_i\colon 1 \le i \le 4,\ 6 \le i \le 9,\ 13 \le i \le 14,\ 18 \le i \le 19,\ 32 \le i \le 34,\ i = 21, 25, 36, 38, 42\}.$$

It follows that

$$\mathrm{k}(G, B_0, d) = \mathrm{k}(N(C(4))), B_0, d) = \begin{cases} 20 & \text{if } d = 3, \\ 6 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\sum_{i=1}^{4}(-1)^{|C(i)|}\,\mathrm{k}(N(C(i)), B_0, d) = 0$ for all integers d and (4A) follows by (3A) when $p = 5$.

Suppose $p = 3$. The proof of (4A) is similar to the case when $p = 5$. Let $C = C(2)$ and $C' = C(3)$ given by (3.1). By (3B) (b), $N(C) \simeq S_3 \times L_2(8) : 3$ and $N(C') \simeq S'.2^2$ and by CAYLEY, the degrees of characters in $\operatorname{Irr}(N(C))$ and $\operatorname{Irr}(N(C'))$ are given by Tables 2 and 3.

Table 2. The degrees of characters of $\operatorname{Irr}(S_3 \times L_2(8) : 3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}
1	1	1	1	1	1	2	2	2	7	7	7	7	7	7	8	8
ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	
8	8	8	8	14	14	14	16	16	16	21	21	27	27	42	54	

Table 3. The degrees of characters of $\operatorname{Irr}(S'.2^2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	1	1	1	1	1	1	1	1	2	2	2
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
2	2	2	2	2	2	2	2	2	4	4	4	6	6	12

A proof similar to that of the case when $p = 5$ shows that the affirmative answer to Dade's conjecture for the cyclic blocks implies that

$$\operatorname{Irr}(N(C), B_1, 1) = \{\xi_{30}, \xi_{31}, \xi_{33}\} \subseteq \operatorname{Irr}(N(C)),$$

and $\operatorname{Irr}(B_0(N(C))) = \operatorname{Irr}(N(C)) \setminus \operatorname{Irr}(N(C), B_1, 1)$. In addition, by CAYLEY, $C_G(S') = Z(S') \le S' \trianglelefteq N(C')$, so that $\operatorname{Irr}(N(C')) = \operatorname{Irr}(B_0(N(C')))$. It follows by the definitions (1.3) and (5.5) of [7] that

$$\mathrm{k}(N(C(2)), B_0, d) = \mathrm{k}(N(C(3)), B_0, d) = \begin{cases} 27 & \text{if } d = 4, \\ 3 & \text{if } d = 3, \\ 0 & \text{otherwise.} \end{cases}$$

By CAYLEY, the degrees of irreducible characters of $N(C(5)) \simeq S\colon (2 \times SD_{2^4})$ and $N(C(6)) \simeq 3^5\colon (2 \times M_{11})$ are given as follows:

Table 4. The degrees of characters of $\mathrm{Irr}(S\colon (2 \times SD_{2^4}))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	1	1	1	1	2	2	2	2	2	2	4	4	4	4
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
4	4	4	4	8	8	8	8	8	8	16	16	16	16	18	18	18	18
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}		
32	36	36	36	72	72	72	72	72	72	72	72	72	72	72	72		

Table 5. The degrees of characters of $\mathrm{Irr}(3^5\colon (2 \times M_{11}))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	10	10	10	10	10	10	11	11	16	16	16	16
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}
22	22	44	44	45	45	55	55	198	198	220	220	220	220
ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}					
220	220	220	352	440	880	880	880	880					

By (2B), $C_G(S) = Z(S) \leq S \trianglelefteq N(C(5))$ and $C_G(3^5) = 3^5 \trianglelefteq N(C(6))$, so that both normalizers $N(C(5))$ and $N(C(6))$ have only the principal block. It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 33 & \text{if } d = 7, \\ \alpha & \text{if } d = 5, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(5), C(6)\}$ and $\alpha = 19$ or 4 according as $C = C(5)$ or $C(6)$. Finally, consider $C = C(4)$. By CAYLEY, the degrees of irreducible characters of $N(C) \simeq 3_+^{1+4}\colon 4S_6$ are given by Table 6.

Table 6. The degrees of characters of $\mathrm{Irr}(3_+^{1+4}\colon 4S_6)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	5	5	5	5	5	5	5	5	8	8	9	9	9	9
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
10	10	10	10	16	16	16	16	18	18	20	20	72	72	72	72	80	80
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}	ξ_{53}	ξ_{54}
80	80	90	90	90	90	160	160	160	160	162	162	180	180	288	288	320	360

Similarly, since $C_G(3_+^{1+4}) = Z(3_+^{1+4})$, it follows that

$$\mathrm{Irr}(N(C(4))) = \mathrm{Irr}(B_0(N(C(4)))).$$

It follows by [6, p. 134] and Table 6 that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 33 & \text{if } d = 7, \\ \beta & \text{if } d = 5, \\ 2 & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(1), C(4)\}$ and $\beta = 4$ or 19 according as $C = C(1)$ or $C(4)$. Thus

$$\sum_{j=1}^{6} (-1)^{|C(j)|}\,\mathrm{k}(N(C(j)), B_0, d) = 0$$

for all integers d, and by (3B) (a), Dade's conjecture holds when $p = 3$.

(4B): *Let B be a 2-block of the simple Conway group $G = \mathrm{Co}_3$ with defect $\mathrm{d}(B) \geq 1$. Then B satisfies the ordinary conjecture of Dade.*

Proof: (1) Let $C(i)$ be given by (3.2). If $b^G = B_2$ for some $b \in \mathrm{Blk}(N(C(i)))$, then by Lemma 6.9 of [7], the final subgroup of $C(i)$ is a conjugate to a subgroup of 2^3, so that $i \in \{1, 10, 13, 16\}$ as 2^3 is $2B$-pure. Let $C = C(13)$ and $C' = C(16)$, so that by (3C) (b), $N(C) \simeq 2 \times A_4 \times S_3$ and $N(C') \simeq (2^3 \times S_3).F_7^3$. Since $N_G(2^3) = N(C')$, it follows by Brauer's First and Third Main Theorems, Theorems III.9.7 and V.5.4 of [8] that $N(C')$ has two blocks $b_0(C')$ and $b_2(C')$ such that $b_0(C')^G = B_0$ and $b_2(C')^G = B_2$. Since every defect group contains $O_2(N(C')) = 2^3$, it follows that $N(C')$ has only two blocks. Now A_4 has exactly one block and S_3 has exactly two blocks, so that $N(C)$ has exactly two blocks $b_0(C)$ and $b_2(C)$ such that $b_0(C) = B_0(N(C))$. By (5.10) of [7] and Brauer's Third Main Theorem, $b_0(C)^G = B_0$ and $b_2(C)^G = B_2$. Let $\mathrm{Irr}^d(B)$ be the subset of $\mathrm{Irr}(B)$ consisting of characters of defect d. By (1.3), $\mathrm{Irr}(N(C), B_i, d) = \mathrm{Irr}^d(b_i(C))$ and $\mathrm{Irr}(N(C'), B_i, d) = \mathrm{Irr}^d(b_i(C'))$ for $i = 0, 2$.

Since $b_0(C) = B_0(2 \times A_4) \times B_0(S_3)$ and $b_2(C) = B_0(2 \times A_4) \times b_2$ for the non-principal block b_2 of S_3, it follows that $|\mathrm{Irr}^d(b_0(C))| = 16$ or 0 according as $d = 4$ or $d \neq 4$, and $|\mathrm{Irr}^d(b_2(C))| = 8$ or 0 according as $d = 3$ or $d \neq 3$. Using CAYLEY, we can get the character table of $N(C')$ and the degrees of characters in $\mathrm{Irr}(N(C'))$ are given by Table 7.

Table 7. The degrees of characters of $\mathrm{Irr}((2^3 \times S_3).F_7^3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	2	2	2	3	3	3	3	6
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}				
6	7	7	7	7	7	7	14	14	14				

Using the method of central characters, we can get the irreducible characters of $b_0(C')$ and $b_2(C')$,

$$\mathrm{Irr}(b_2(C')) = \{\xi_7, \xi_8, \xi_9, \xi_{14}, \xi_{15}, \xi_{22}, \xi_{23}, \xi_{24}\}$$

and $\mathrm{Irr}(b_0(C')) = \mathrm{Irr}(N(C')) \setminus \mathrm{Irr}(b_2(C'))$. It follows that

$$\mathrm{k}(N(C(13)), B, d) = \mathrm{k}(N(C(16)), B, d) = \begin{cases} \alpha & \text{if } d = 4, \\ \beta & \text{if } d = 3, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\alpha, \beta) = (16, 0)$ or $(0, 8)$ according as $B = B_0$ or B_2. Since $N(C(10)) \simeq 2^* \times M_{12}$, it follows by James [10, Theorem 8.2] and (2D) that $\mathrm{k}(G, B_2, d) = \mathrm{k}(N(C(10)), B_2, d) = 8$ or 0 according as $d = 3$ or $d \neq 3$. This proves (4B) when $B = B_2$.

(2) Suppose $B = B_0$. By CAYLEY, the degrees of irreducible characters of $N(C(9)) \simeq 2^3.2^3.S_3$ and $N(C(14)) \simeq 2^3.2^3.2$ are given by Tables 8 and 9.

Table 8. The degrees of characters of $\mathrm{Irr}(2^3.2^3.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	1	1	1	1	2	2	2	2	3	3
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}
3	3	3	3	3	3	6	6	6	6	6	6	6	6

Table 9. The degrees of characters of $\mathrm{Irr}(2^3.2^3.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}
2	2	2	2	2	2	2	2	2	2	2	2	4	4	4	4

Since $C_G(2^3.2^3) = 2 \times C_{M_{12}}(E_4.E_8) = Z(2^3.2^3)$ (cf. [3, (2D)]), it follows by Corollary V.3.11 of [8] that $N(C(9))$ has the unique block $B_0(N(C(9)))$. Similarly, $B_0(N(C(14)))$ is the unique block of $N(C(14))$. It follows by (1.3) that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 7, \\ 12 & \text{if } d = 6, \\ \gamma & \text{if } d = 5, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(9), C(14)\}$ and $\gamma = 0$ or 4 according as $C = C(9)$ or $C(14)$. By CAYLEY, the degrees of irreducible characters of $N(C(11)) \simeq 2 \times 2_+^{1+4}.S_3$ are given by the following table:

Table 10. The degrees of characters of $\mathrm{Irr}(2 \times 2_+^{1+4}.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	1	1	1	2	2	3	3	3	3	3	3	3	3
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}		
3	3	3	3	4	4	4	4	6	6	8	8		

Since $C_G(2 \times 2_+^{1+4}) = 2 \times C_{M_{12}}(2_+^{1+4}) = Z(2 \times 2_+^{1+4})$, it follows that the principal block is the only block of $N(C(11))$. By (3C) (b), $N(C(10)) = 2^* \times M_{12}$ and the principal block of M_{12} is given by James [10, Theorem 8.2]. If $C = C(10)$ or $C(11)$, then by Table 10 and [10, Theorem 8.2],

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 7, \\ 4 & \text{if } d = 6, \\ \eta & \text{if } d = 5, \\ 2 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $\eta = 0$ or 4 according as $C = C(10)$ or $C(11)$. Similarly, $B_0(N(C(15)))$ is the only block of $N(C(15))$ and by CAYLEY, $N(C(15)) \simeq (2^4)^*.F_7^3$ has 16 irreducible characters of height 0. Since $N(C(12)) = 2^2 \times A_4$, the principal block of $N(C(12))$ is the only block of $N(C(12))$ and moreover, $\mathrm{k}(N(C(15)), B_0, d) = \mathrm{k}(N(C(12)), B_0, d) = 16$ or 0 according as $d = 4$ or $d \neq 4$. It follows that

$$\sum_{i=9}^{16} (-1)^{|C(i)|}\, \mathrm{k}(N(C(i)), B_0, d) = 0. \tag{4.1}$$

By CAYLEY, the degrees of irreducible characters of $N(C(3)) \simeq 2^2.2^6.F_{3^2}^2.2$ and $N(C(4)) \simeq 2^2.2^6.F_{3^2}^2.S_3$ are given by Tables 11 and 12.

Table 11. The degrees of characters of $\mathrm{Irr}(2^2.2^6.F_{3^2}^2.2)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}
1	1	1	1	2	2	2	2	3	3	3	3	4	4	4	4	4	6
ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}	ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}
6	6	6	8	8	8	8	8	8	9	9	9	9	9	9	9	9	12
ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}	ξ_{46}	ξ_{47}	ξ_{48}	ξ_{49}	ξ_{50}	ξ_{51}	ξ_{52}	ξ_{53}	ξ_{54}
12	12	12	12	16	16	16	16	16	18	18	18	18	24	24	24	24	48

Table 12. The degrees of characters of $\mathrm{Irr}(2^2.2^6.F_{32}^2.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	2	2	4	6	6	9	9	9	9
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
9	9	9	9	12	12	12	12	18	18	18	18	18	18	24
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}
24	24	24	27	27	27	27	36	36	36	36	36	48	72	72

Since $C_G(2^2.2^6) = Z(2^2.2^6) \leq 2^2.2^6$, it follows by [8, Corollary V.3.11] that the principal block is the only block of $N(C(3))$ and $N(C(4))$. If $C = C(3)$ or $C(4)$, then

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 12 & \text{if } d = 9, \\ 10 & \text{if } d = 8, \\ \alpha & \text{if } d = 7, \\ \beta & \text{if } d = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where $(\alpha, \beta) = (10, 6)$ or $(6, 1)$ according as $C = C(3)$ or $C = C(4)$. By (2C), $N(C(5)) \simeq 2_+^{1+6}.L_3(2)$ and $N(C(6)) \simeq 2^4.A_8$ and by CAYLEY, the degrees of irreducible characters of $N(C(6))$ are given by Table 13.

Table 13. The degrees of characters of $\mathrm{Irr}(2^4.A_8)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}
1	7	14	15	20	21	21	21	28	35	45	45	45	45
ξ_{15}	ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}			
56	64	70	90	105	105	105	120	210	315	315			

The degrees of irreducible characters of $2_+^{1+6}.L_3(2)$ are given by [2, Table X], and moreover, both $N(C(5))$ and $N(C(6))$ have exactly one block, the principal block. It follows by Table 13 and [2, Table X] that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 4 & \text{if } d = 9, \\ 2 & \text{if } d = 8, \\ x & \text{if } d = 7, \\ y & \text{if } d = 6, \\ 1 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(5), C(6)\}$ and $(x, y) = (6,1)$ or $(2,0)$ according as $C = C(5)$ or $C(6)$.

By CAYLEY, the degrees of irreducible characters of $N(C(7)) \simeq 2^2.2^6.S_3.S_3$ and $N(C(8)) \simeq 2.2^3.2^5.S_3$ are given by Tables 14 and 15.

Table 14. The degrees of characters of $\mathrm{Irr}(2^2.2^6.S_3.S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	2	2	3	3	3	3	4	6	6
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
6	6	9	9	9	9	9	9	9	9	12	12	12	12	12
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}					
18	18	18	18	24	24	36	36	36	36					

Table 15. The degrees of characters of $\mathrm{Irr}((2.2^3.2^5).S_3)$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	1	1	1	2	2	3	3	3	3	3	3	3	3	3
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
3	3	3	4	4	4	4	6	6	6	6	6	6	6	6
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}	ξ_{44}	ξ_{45}
6	6	8	8	8	8	12	12	12	12	12	12	16	24	24

Moreover, the principal block is the only block of $N(C(7))$ and $N(C(8))$. It follows that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 12 & \text{if } d = 9, \\ 10 & \text{if } d = 8, \\ u & \text{if } d = 7, \\ v & \text{if } d = 6, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(7), C(8)\}$, $(u, v) = (2,0)$ or $(6,1)$ according as $C = C(7)$ or $C(8)$.

By CAYLEY, the degrees of irreducible characters of $N(C(2)) \simeq 2.S_6(2)$ are given by Table 16.

Table 16. The degrees of characters of $\mathrm{Irr}(2.S_6(2))$

ξ_1	ξ_2	ξ_3	ξ_4	ξ_5	ξ_6	ξ_7	ξ_8	ξ_9	ξ_{10}	ξ_{11}	ξ_{12}	ξ_{13}	ξ_{14}	ξ_{15}
1	7	8	15	21	21	27	35	35	48	56	64	64	70	84
ξ_{16}	ξ_{17}	ξ_{18}	ξ_{19}	ξ_{20}	ξ_{21}	ξ_{22}	ξ_{23}	ξ_{24}	ξ_{25}	ξ_{26}	ξ_{27}	ξ_{28}	ξ_{29}	ξ_{30}
105	105	105	112	112	120	120	168	168	189	189	189	210	210	216
ξ_{31}	ξ_{32}	ξ_{33}	ξ_{34}	ξ_{35}	ξ_{36}	ξ_{37}	ξ_{38}	ξ_{39}	ξ_{40}	ξ_{41}	ξ_{42}	ξ_{43}		
280	280	280	315	336	378	405	420	448	512	512	560	720		

By Lemma 6.9 of Dade [7], $\mathrm{k}(N(C), B_1, d) \geq 1$ for some radical chain C implies that the final subgroup of C is conjugate to some defect group of $D(B_1) = 2$. Thus $C =_G C(2)$ or $C(1)$. Since Dade's conjecture for the cyclic blocks has an affirmative answer, it follows that $\mathrm{k}(N(C(1)), B_1, d) = \mathrm{k}(N(C(2)), B_1, d)$ for all integers d. In particular, by (2D) (b), $\mathrm{Irr}(N(C(2)), B_1, d) = 2$ or 0 according as $d = 1$ or $d \neq 1$. Since $N(C(2))$ has exactly two irreducible characters ξ_{40} and ξ_{41} of defect 1, it follows that $\mathrm{Irr}(N(C(2)), B_1, 1) = \{\xi_{40}, \xi_{41}\}$. Since $N(C(2)) = N(D(B_1))$, it follows by Brauer's First Main Theorem that $N(C(2))$ has a unique block b_1 inducing B_1. Thus $\mathrm{Irr}(b_1) = \{\xi_{40}, \xi_{41}\}$, since each character of $\mathrm{Irr}(b_1)$ is a character of $\mathrm{Irr}(N(C(2)), B_1, d)$ for some d. This implies that $\mathrm{Irr}(B_0(N(C(2)))) = \mathrm{Irr}(N(C(2)))\setminus \mathrm{Irr}(b_1)$. It follows by Table 16 and [6, p.135] that

$$\mathrm{k}(N(C), B_0, d) = \begin{cases} 16 & \text{if } d = 10, \\ 4 & \text{if } d = 9, \\ 2 & \text{if } d = 8, \\ a & \text{if } d = 7, \\ b & \text{if } d = 6, \\ 3 & \text{if } d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

where $C \in \{C(1), C(2)\}$ and $(a, b) = (6, 1)$ or $(10, 6)$ according as $C = C(1)$ or $C(2)$. Thus (4B) follows by (3C) (a), (4.1) and

$$\sum_{j=1}^{8} (-1)^{|C(j)|}\, \mathrm{k}(N(C(j)), B_0, d) = 0.$$

This completes the proof.

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